PMM U.S.S.R.,Vol.49,5,pp.608-615,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

RELAXATION-TYPE EQUATIONS FOR VISCOELASTIC MEDIA WITH FINITE DEFORMATIONS*

V.I. KONDAUROV

The representation of the total deformations in the form of a composition of elastic and irreversible components is considered to describe finite deformations of viscoelastic materials of relaxation type. According to the method of internal parameters, it is assumed that the thermodynamic potential, the stress tensor, the entropy density, the thermal flux, and the rate of change of inelastic deformations are functions of the total deformation, the temperture, the temperature gradient, and the irreversible deformation. On the basic of requirements for invariance of the governing equations, a definition of isotropic ideal and hardened viscoelastic bodies is given. Necessary and sufficient conditions are formulated which the equations of state of such media should satisfy. The propagation of isothermal waves of weak discontinuity in an ideal viscoelastic medium with small elastic and finite total deformations is considered as an illustration.

The governing relationships of relaxation-type considered below occupy an intermediate location between the equations of media with infinitesimal memory and equations of state with weakly damped memory of general form in the degree of generality /1/. The simplest equation of this kind is the one-dimensional Maxwell equation. Its distinctive generalizations, a survey of which can be found in /2—4/, reduce in purely mechanical theory mainly to the consideration of the spatial state of stress and strain and the introduction of time derivatives of the stress and strain tensors of order higher than the first. For finite deformations the problem arises of selecting the preferable form of the objective derivatives.

An approach based on introducing generalized Maxwell's equations for not only the stresses but also for the other rheological relationships of a thermodynamic nature has not been applied extensively in the thermomechanics of viscoelastic media of relaxation type. The method of latent, or internal, parameters /5/ has turned out to be more general and fruitful. In conformity with this method, the running state of a material particle is described not only by the deformations, temperature, and temperature gradient but also by the internal parameters. A system of additional rheological relationships are introduced for the latter. As a rule, these relationships are ordinary differential equations with initial data. Integration of the equations for the internal parameters for given prehistories of the deformation, temperature, and temperature gradient show that all the rheological characteristics are functionals of the deformation process, but functionals of a particular kind governed by the solution of the above-mentioned problem with the initial data.

A set of N tensors of the inelastic components of the gradient of a non-degenerate mapping of the reference configuration of the body into a real configuration is taken in this paper as the internal parameters of the state of the medium. Such an approach, in which the gradient of the mapping is represented as the product of the instantaneous elastic and N inelastic components is a generalization of the expansion that is utilized extensively to describe kinematically elastic-plastic media with finite deformations /6—8/. It enables one to take into account phenomena that are characterized, in the linear case, by the spectrum of N relaxation times.

Furthermore, without relying on assumptions of a particular nature (such as the assumption that the deformations are small), the constraints imposed on the governing relations by the inequality of the entropy and invariance requirements are studied. In addition to the well-known invariance requirements of the governing equations for the replacement of the reference system (the objectivity principle) and the orthogonal transformations by an undistorted reference configuration (material isotropy) /1/, invariance relative to orthogonal transformations of Euclidean spaces tangent to spaces of instantaneously unloaded intermediate configurations at this point of the body is used in a substantial manner. The latter is none other than the assumption /8/ that the motion as a rigid whole has no influence on the rheological characteristics of a homogeneously deformed and unloaded body.

The requirements mentioned enable us to formulate the necessary and sufficient conditions that the governing equation should satisfy. These invariance conditions certainly do not yield

*Prikl.Matem.Mekhan.,49,5,791-800,1985

a unique answer to the question as to what is the specific form of the functions in the equations, but substantially narrow down the class of allowable equations of state. This is particularly so when there are additional constraints, as is illustrated in the example of an isothermal ideal viscoelastic medium with small elastic but finite total deformations.

1. Kinematics. Let $\mathbf{X}$ be the radius-vector of a material particle of a body in a reference configuration (RC) $\varkappa$, $\mathbf{x}$ in a real configuration (AC) χ, corresponding to the running time t, such that

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \quad d\mathbf{x} = \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}, \quad \Delta = \det \mathbf{F} \neq 0 \tag{1.1}$$

The tensor $\mathbf{F}$ is the gradient of the non-degenerate mapping (1.1), and the vector $\mathbf{v} = \partial\mathbf{x}(\mathbf{X}, t)/\partial t$ defines the velocity of the material particle.

If the mapping (1.1) is twice continuously differentiable, then a local conservation law holds for the compatibility of the velocity and total deformation fields which is written in the variables $\mathbf{X}$, t in the form ($\mathbf{I}$ is the unit tensor)

$$\mathbf{F}^{\cdot} - \mathrm{Div}\,(\mathbf{v} \otimes \mathbf{I}) = 0 \tag{1.2}$$

In addition to the RC and AC of the body we introduce N intermediate configurations (IC) $\varkappa_\alpha$, $\alpha = 1, 2, \ldots, N$, $N \geqslant 1$. The particle radius-vector $\mathbf{X}$ in these configurations equals

$$\mathbf{X}_\alpha = \mathbf{X}_\alpha(\mathbf{X}_{\alpha-1}, t), \quad \mathbf{X}_0 \equiv \mathbf{X}, \ \alpha = 1, 2, \ldots, N$$

We consider the mapping $\varkappa_{\alpha-1} \to \varkappa_\alpha$ as one-to-one and sufficiently smooth. We denote the gradients of the mapping $\varkappa_{\alpha-1} \to \varkappa_\alpha$ by $\mathbf{P}_\alpha$ so that

$$d\mathbf{X}_\alpha = \mathbf{P}_\alpha\, d\mathbf{X}_{\alpha-1}, \quad \det \mathbf{P}_\alpha \neq 0 \tag{1.3}$$

and we call the gradients inelastic mappings.

Here and henceforth there is no summation over α if not specially stipulated.

In contrast to the RC and AC, the IC of a body belongs to a non-Euclidean space in the general case. Consequently, the tensors of the second rank $\mathbf{P}_\alpha$ should be considered as the mapping of Euclidean spaces tangent at a given point $\mathbf{X}$ to three-dimensional spaces containing the configurations $\varkappa_{\alpha-1}$ and $\varkappa_\alpha$. The tangential Euclidean space can be treated as a space in which the IC of a homogeneously deformed body would be found with deformations equal everywhere to the deformations at the point $\mathbf{X}$.

If $\mathbf{E}$ denotes the gradient of the non-degnerate mapping $\varkappa_N \to \chi$, which we shall call the gradient of an elastic mapping, then it follows from (1.1) and (1.3) that

$$\mathbf{F} = \mathbf{E}\mathbf{P}_N\mathbf{P}_{N-1} \ldots \mathbf{P}_2\mathbf{P}_1 \tag{1.4}$$

The introduction of the IC $\varkappa_\alpha$ and the representation (1.4) of the gradient $\mathbf{F}$ in the form of gradients of an elastic and irreversible mapping is used extensively to construct models of elastic-plastic bodies with finite deformations /6–8/. As in the theory of plasticity, the expansion (1.4) does not govern the order of the elastic and viscous deformation processes in time, that are simultaneously developed physically in the body.

Since all the mappings utilized are non-singular, then by using the theorem on the polar decomposition of a tensor of the second rank, we can write

$$\mathbf{F} = \mathbf{R}\mathbf{U}, \quad \mathbf{E} = \mathbf{R}_e\mathbf{U}_e, \quad \mathbf{P}_\alpha = \mathbf{H}_\alpha\mathbf{W}_\alpha \tag{1.5}$$

where $\mathbf{R}$, $\mathbf{R}_e$, $\mathbf{H}_\alpha$ are orthogonal and $\mathbf{U}$, $\mathbf{U}_e$, $\mathbf{W}_\alpha$ are symmetric positive-definite tensors.

2. Governing relationships. We will consider homogeneous thermoviscoelastic materials of relaxation type. It is assumed that the state of a material particle $\mathbf{X}$ at the time t is given completely if the following set of quantities is known

$$\lambda(\mathbf{X}, t) = \{\mathbf{F}(\mathbf{X}, t), \mathbf{P}_\alpha(\mathbf{X}, t), \theta(\mathbf{X}, t), \boldsymbol{\gamma}(\mathbf{X}, t)\} \tag{2.1}$$

where $\boldsymbol{\gamma}$ is the gradient of the scalar field of the absolute temperature $\theta > 0$, so that $d\theta = \boldsymbol{\gamma} d\mathbf{X}$. We will call the family of states parametrically dependent on t a generalized process of particle deformation.

The governing of rheological equations of the materials under consideration are finite, non-differential relationships

$$A = A^+(\lambda), \quad \mathbf{T} = \mathbf{T}^+(\lambda), \quad \eta = \eta^+(\lambda), \quad \mathbf{q} = \mathbf{q}^+(\lambda) \tag{2.2}$$

and the evolutionary equations

$$\boldsymbol{\Phi}_\alpha(\mathbf{P}_\alpha^{\cdot}, \lambda) = 0, \quad \alpha = 1, 2, \ldots, N \tag{2.3}$$

The quantities A, η in relationships (2.2) are the free energy density and the entropy, $\mathbf{T}$ is the Cauchy stress tensor, and $\mathbf{q}$ is the thermal flux vector. The tensor functions of second rank $\boldsymbol{\Phi}_\alpha$ relating the rate of change of the gradients $\mathbf{P}_\alpha$ to the running state of the particle govern the equations for the rate of production of the inelastic deformations.

Let $\mathbf{K} = \mathrm{const}$ be the gradient of unimodular transformations of one RC $\varkappa$ into another

RC $\varkappa'$. If the governing equations (2.2), (2.3) do not change form during the mapping $\varkappa \to \varkappa'$, then such two configurations are called equitable while the mapping $\varkappa \to \varkappa'$ is the equitability transformation /1, 9/. Taking into account that on replacing the reference configuration

$$\mathbf{F} \to \mathbf{FK}, \mathbf{P}_1 \to \mathbf{P}_1\mathbf{K}, \mathbf{P}_\beta \to \mathbf{P}_\beta, \quad \mathbf{P}_1^{\cdot} \to \mathbf{P}_1^{\cdot}\mathbf{K}, \quad \gamma \to \mathbf{K}^T\gamma$$
$$\beta = 2, 3, \ldots, N$$

we find that the equitability conditions for $\varkappa$ and $\varkappa'$ are the relationships

$$A^+(\mathbf{F}, \mathbf{P}_\beta, \mathbf{P}_1, \theta, \gamma) = A^+(\mathbf{FK}, \mathbf{P}_\beta, \mathbf{P}_1\mathbf{K}, \theta, \mathbf{K}^T\gamma), \ldots, \mathbf{\Phi}_1(\mathbf{P}_1^{\cdot}\mathbf{K}, \cdot) = 0 \qquad (2.4)$$
$$\mathbf{\Phi}_\beta(\mathbf{P}_\beta^{\cdot}, \cdot) = 0$$

The gradients of the equtability transformation generate a group that we denote by $g_\varkappa$.

Let $\mathbf{Q} = \mathbf{Q}(t)$ be an orthogonal tensor which is the gradient of the transformation of one AC χ into another AC χ' for fixed RC and IC. Under such transformations

$$\mathbf{F} \to \mathbf{QF}, \quad \mathbf{P}_\alpha \to \mathbf{P}_\alpha, \mathbf{T} \to \mathbf{QTQ}^T, \quad \mathbf{q} \to \mathbf{Qq}, \quad \theta \to \theta, \gamma \to \gamma$$

and by virtue of the principle of objectivity /1, 9/, the following equations hold for values of the governing mappings (2.2), (2.3)

$$A^+(\mathbf{QF}, \mathbf{P}_\alpha, \theta, \gamma) = A^+(\mathbf{F}, \cdot), \mathbf{T}^+(\mathbf{QF}, \cdot) = \mathbf{QT}^+(\mathbf{F}, \cdot)\,\mathbf{Q}^T \qquad (2.5)$$
$$\eta^+(\mathbf{QF}, \cdot) = \eta^+(\mathbf{F}, \cdot), \quad \mathbf{q}^+(\mathbf{QF}, \cdot) = \mathbf{Qq}^+(\mathbf{F}, \cdot),$$
$$\mathbf{\Phi}_\alpha(\mathbf{P}_\alpha^{\cdot}, \mathbf{QF}, \cdot) = 0; \quad \alpha = 1, 2, \ldots, N$$

We will now consider the transformation of any intermediate configuration $\varkappa_\alpha$. If $\mathbf{Z}_\alpha = \mathbf{Z}_\alpha(\mathbf{X}, t)$ $(\det \mathbf{Z}_\alpha = 1)$ is the gradient of the unimodular transformation $\varkappa_\alpha \to \varkappa_\alpha'$ then for the remaining fixed configurations the connection between the arguments (2.1) of the governing relationships has the form

$$\mathbf{F} \to \mathbf{F}, \quad \mathbf{P}_\beta \to \mathbf{P}_\beta, \quad \mathbf{P}_\alpha \to \mathbf{Z}_\alpha\mathbf{P}_\alpha, \quad \mathbf{P}_{\alpha+1} \to \mathbf{P}_{\alpha+1}\mathbf{Z}_\alpha^{-1}, \quad \theta \to \theta$$
$$\gamma \to \gamma; \quad \beta = 1, 2, \ldots, N, \quad \beta \neq \alpha, \alpha + 1$$

As before for the reference, for the intermediate configurations we introduce the definition of equitability, in conformity with which two intermediate configurations $\varkappa_\alpha$ and $\varkappa_\alpha'$ are equitable if the transformation $\varkappa_\alpha \to \varkappa_\alpha'$ leaves unchanged the value of the thermodynamic potential, the Cauchy stress tensor, the entropy, end the thermal flux and the form of the evolutionary equations does not change, i.e.,

$$A^+(\mathbf{F}, \mathbf{P}_\beta, \mathbf{Z}_\alpha\mathbf{P}_\alpha, \mathbf{P}_{\alpha+1}\mathbf{Z}_\alpha^{-1}, \theta, \gamma) = A^+(\mathbf{F}, \mathbf{P}_\beta, \mathbf{P}_\alpha, \mathbf{P}_{\alpha+1}, \theta, \gamma), \ldots, \qquad (2.6)$$
$$\mathbf{\Phi}_\beta(\mathbf{P}_\beta^{\cdot}, \mathbf{F}, \mathbf{P}_\beta, \mathbf{Z}_\alpha\mathbf{P}_\alpha, \mathbf{P}_{\alpha+1}\mathbf{Z}_\alpha^{-1}, \theta, \gamma) = 0,$$
$$\mathbf{\Phi}_\alpha(\mathbf{Z}_\alpha^{\cdot}\mathbf{P}_\alpha + \mathbf{Z}_\alpha\mathbf{P}_\alpha^{\cdot}, \cdot) = 0, \quad \mathbf{\Phi}_{\alpha+1}(\mathbf{P}_{\alpha+1}\mathbf{Z}_\alpha^{-1^{\cdot}} + \mathbf{P}_{\alpha+1}^{\cdot}\mathbf{Z}_\alpha^{-1}, \cdot) = 0$$

The transformation gradients $\varkappa_\alpha$ in the equitable configurations $\varkappa_\alpha', \varkappa_\alpha'', \ldots$ generate a group which we denote by $g_{\varkappa_\alpha}$.

We define an isotropic viscoelastic body of relaxation type as a material in which the reference configuration $\varkappa_0$ exists with an equitability group that contains the complete orthogonal group o; the latter also belongs to the equitability groups of all the intermediate configurations. In other words, in the case of an isotropic material

$$o \in g_{\varkappa_0}, \ o \in g_{\varkappa_\alpha}, \ \alpha = 1, 2, \ldots, N$$

Since $o \in g_{\varkappa_0} \in u$, u is a unimodular group, then either $g_{\varkappa_0} = o$ or $g_{\varkappa_0} = u$ as is shown in /10/. Analogously, $g_{\varkappa_\alpha} = o$ or $g_{\varkappa_\alpha} = u$. Consequently, even in the simplest case when $N = 1$ and there is one IC, four kinds of materials are possible, governed by the relationships

$$g_{\varkappa_0} = g_{\varkappa_1} = o; \quad g_\varkappa = u, g_{\varkappa_1} = o; \quad g_{\varkappa_0} = o, \quad g_{\varkappa_1} = u; \quad g_\varkappa = g_{\varkappa_1} = u$$

We consider two viscoelastic materials of relaxation type. The first is determined by the condition

$$g_{\varkappa_0} = g_{\varkappa_\alpha} = o, \quad \alpha = 1, 2, \ldots, N \qquad (2.7)$$

and will be called a hardening viscoelastic body. It is assumed for the second that

$$g_\varkappa = g_{\varkappa_\beta} = u, \ g_{\varkappa_N} = o, \ \beta = 1, 2, \ldots, N - 1 \qquad (2.8)$$

This material will be called an ideal viscoelastic body.

For a hardening viscoelastic body(2.7) it is necessary and sufficient that the governing equations have the form

$$A = A^+(\lambda_1), \quad \mathbf{T} = \mathbf{RT}^+(\lambda_1)\,\mathbf{R}^T, \quad \eta = \eta^+(\lambda_1), \quad \mathbf{q} = \mathbf{Rq}^+(\lambda_1) \qquad (2.9)$$
$$\mathbf{V}_\alpha^{\cdot} = \mathbf{\Psi}_\alpha(\lambda_1)$$

and that the following isotropy properties be satisfied:

$$A^+(\lambda_1) = A^+(\lambda_1^Q), \quad \mathbf{QT}^+(\lambda_1)\,\mathbf{Q}^T = \mathbf{T}^+(\lambda_1^Q), \qquad (2.10)$$
$$\eta^+(\lambda_1) = \eta^+(\lambda_1^Q), \quad \mathbf{Qq}^+(\lambda_1) = \mathbf{q}^+(\lambda_1^Q), \quad \mathbf{Q\Psi}_\alpha(\lambda_1)\,\mathbf{Q}^T = \mathbf{\Psi}_\alpha(\lambda_1^Q)$$

Here

$$\lambda_1(\mathbf{X},t)=\{\mathbf{U},\mathbf{V}_\alpha,\theta,\gamma\}$$
$$\lambda_1{}^Q=\{\mathbf{QUQ}^T,\mathbf{QV}_\alpha\mathbf{Q}^T,\theta,\mathbf{Q}\gamma\}$$
$$\mathbf{V}_1\equiv\mathbf{W}_1,\quad \mathbf{V}_\alpha=(\mathbf{H}_{\alpha-1}\mathbf{H}_{\alpha-2}\ldots\mathbf{H}_1)^T\mathbf{W}_\alpha(\mathbf{H}_{\alpha-1}\mathbf{H}_{\alpha-2}\ldots\mathbf{H}_1)$$
$$\alpha=2,3,\ldots,N \tag{2.11}$$

To prove the necessity of (2.9) and (2.10) we ocnsider the invariance conditions (2.4) with respect to the constant orthogonal transformation of an undistorted IC. Assuming the transformation gradient $\mathbf{K}=\mathbf{R}^T(\mathbf{X}_0,\ t_0)$, $\mathbf{X}_0=\mathrm{const}$, $t_0=\mathrm{const}$, we obtain

$$A=A^+(\mathbf{F},\mathbf{P}_\beta,\mathbf{P}_1,\theta,\gamma)=A^+(\mathbf{RUR}^T,\mathbf{P}_\beta,\mathbf{P}_1\mathbf{R}^T,\theta\ \mathbf{R}\gamma),\ldots, \tag{2.12}$$
$$\mathbf{\Phi}_1(\mathbf{P}_1{}^{\cdot}\mathbf{R}^T,\cdot)=\theta,\ \mathbf{\Phi}_\beta(\mathbf{P}_\beta{}^{\cdot},\cdot)=0$$
$$\beta=2,3,\ldots,N$$

Now let $\mathbf{Z}_1(\mathbf{X},\ t)=\mathbf{R}(\mathbf{X},\ t_0)\,\mathbf{H}_1^T(\mathbf{X},\ t)(t\leqslant t_0)$ be a time-dependent gradient of the transformation of the Euclidean space tangent to the IC $\varkappa_1$ at the point $\mathbf{X}$. This transformation is orthogonal. Its time-derivative has the form $\mathbf{Z}_1{}^{\cdot}(\mathbf{X},\ t)=\mathbf{R}(\mathbf{X},\ t_0)\,\mathbf{H}_1^{T\cdot}(\mathbf{X},\ t)$. Then conditions (2.6) applied to (2.12) yield

$$A=A^+(\mathbf{RUR}^T,\mathbf{P}_\beta,\mathbf{P}_2\mathbf{H}_1\mathbf{R}^T,\mathbf{RW}_1\mathbf{R}^T,\theta,\mathbf{R}\gamma),,\ldots,\mathbf{\Phi}_1(\mathbf{RW}_1{}^{\cdot}\mathbf{R}^T,\cdot)=0 \tag{2.13}$$
$$\mathbf{\Phi}_2\left(\frac{\partial}{\partial t}(\mathbf{P}_2\mathbf{H}_1)|_X\mathbf{R}^T,\cdot\right)=0,\mathbf{\Phi}_\beta(\mathbf{P}_\beta^{\cdot},\cdot)=0,\quad \beta=3,4,\ldots,N$$

Analogously sequentially examining the orthogonal transformations

$$\mathbf{Z}_\alpha(\mathbf{X},\ t)=\mathbf{R}(\mathbf{X},\ t_0)\,\mathbf{H}_{\alpha-1}^T(\mathbf{X},\ t)\,\mathbf{H}_{\alpha-2}^T(\mathbf{X},\ t)\ldots\mathbf{H}_1{}^T(\mathbf{X},\ t)$$

of the Euclidean spaces tangent to the IC $\varkappa_\alpha$ $(\alpha=2,3,\ldots,N)$ at the point $\mathbf{X}$ we arrive at the necessary form of the governing equations

$$A=A^+(\lambda_1{}^R),\quad \mathbf{T}=\mathbf{T}^+(\lambda_1{}^R),\quad \eta=\eta^+(\lambda_1{}^R),\quad \mathbf{q}=\mathbf{q}^+(\lambda_1{}^R)$$
$$\mathbf{\Phi}_\alpha(\mathbf{RV}_\alpha{}^{\cdot}\mathbf{R}^T,\lambda_1{}^R)=0$$

where $\lambda_1{}^R$ is given by (2.11).

If the equations for the rates of production of the inelastic deformations allow of a form solved with respect to the time derivatives, then

$$\mathbf{RV}_\alpha{}^{\cdot}\mathbf{R}^T=\mathbf{\Psi}_\alpha(\lambda_1{}^R)$$

Hence, tkaing the objectivity principle into atcount we obtain (2.9) and (2.10), for $\mathbf{Q}(t)=\mathbf{R}^T(\mathbf{X}_0,\ t)$.

The sufficiency of the forms (2.9), (2.10) of the governing equations for the satisfaction of (2.7) and the validity of the objectivity principle follows from the direct substitution into (2.9) of the formulas

$$\mathbf{F}^*=\mathbf{QFK}=(\mathbf{QRK})(\mathbf{K}^T\mathbf{UK})$$
$$\mathbf{P}_\alpha{}^*=\mathbf{Z}_\alpha\mathbf{P}_\alpha\mathbf{Z}_{\alpha-1}^T=(\mathbf{Z}_\alpha\mathbf{H}_\alpha\mathbf{Z}_{\alpha-1}^T)(\mathbf{Z}_{\alpha-1}\mathbf{W}_\alpha\mathbf{Z}_{\alpha-1}^T),\quad \mathbf{Z}_0\equiv\mathbf{K}$$
$$\mathbf{R}^*=\mathbf{QRK},\quad \mathbf{U}^*=\mathbf{K}^T\mathbf{UK}$$
$$\mathbf{H}_\alpha{}^*=\mathbf{Z}_\alpha\mathbf{H}_\alpha\mathbf{Z}_{\alpha-1}^T,\quad \mathbf{W}_\alpha{}^*=\mathbf{Z}_{\alpha-1}\mathbf{W}_\alpha\mathbf{Z}_{\alpha-1}^T$$

obtained taking the uniqueness of the polar expansion and the definition (2.11) of the tensors $\mathbf{V}_\alpha$ into account.

In the case of ideal viscoelastic media, it is necessary and sufficient for the satisfaction of conditions (2.8) and the objectivity principle, that the governing relationships have the form

$$A=A^+(\lambda_2),\quad \mathbf{T}=\mathbf{RT}^+(\lambda_2)\mathbf{R}^T,\quad \eta=\eta^+(\lambda_2),\quad \mathbf{q}=\mathbf{Rq}^+(\lambda_2) \tag{2.14}$$
$$\mathbf{V}_N{}^{\cdot}\mathbf{V}_N^{-1}=\mathbf{\Psi}_N(\lambda_2),\quad \mathbf{V}_\alpha{}^{\cdot}\mathbf{V}_\alpha^{-1}=\mathbf{V}_{\alpha-1}^{-1}\ldots\mathbf{V}_N^{-1}\mathbf{\Psi}_\alpha(\lambda_2)\mathbf{V}_N\ldots\mathbf{V}_{\alpha+1}$$
$$\alpha=1,2,\ldots,N-1$$

where $A^+,\mathbf{T}^+,\eta^+,\mathbf{q}^+$ and $\mathbf{\Psi}_\alpha$ are isotropic functions, i.e.,

$$A^+(\lambda_2)=A^+(\lambda_2{}^Q),\quad \mathbf{QT}^+(\lambda_2)\mathbf{Q}^T=\mathbf{T}^+(\lambda_2{}^Q), \tag{2.15}$$
$$\eta^+(\lambda_2)=\eta^+(\lambda_2{}^Q),\quad \mathbf{Qq}^+(\lambda_2)=\mathbf{q}^+(\lambda_2{}^Q),\quad \mathbf{Q\Psi}_\alpha(\lambda_2)\mathbf{Q}^T=\mathbf{\Psi}_\alpha(\lambda_2{}^Q)$$

Here

$$\lambda_2(\mathbf{X},t)=\{\mathbf{B},\Delta_\alpha,\theta,\mathbf{B}^{-1T}\nabla\theta\},\quad \lambda_2{}^Q=\{\mathbf{QBQ}^T,\Delta_\alpha,\theta,\mathbf{QB}^{-1T}\nabla\theta\} \tag{2.16}$$
$$\mathbf{B}=\mathbf{UV}_1^{-1}\mathbf{V}_2^{-1}\ldots\mathbf{V}_N^{-1},\quad \Delta_\alpha=\det\mathbf{P}_\alpha=\det\mathbf{V}_\alpha$$

To prove the necessity of (2.14), (2.15), the unimodular transformation of the RC $\varkappa$ with the constant gradient $\mathbf{K}=\Delta^3(\mathbf{X}_0,t_0)\,\mathbf{P}_1{}^{-1}(\mathbf{X}_0,t_0)$, the unimodular transformations of the IC $\varkappa_\beta$ $(\beta=1,2,\ldots,N-1)$ with the constant gradients $\mathbf{Z}_\beta=\Delta_{\beta-1}^3(\mathbf{X}_0,t_0)\,\mathbf{P}_{\beta+1}(\mathbf{X}_0,t_0)$, the orthogonal transformation $\varkappa_N$ with the gradient $\mathbf{Z}_N(t)=\mathbf{R}(t_0)\,\mathbf{H}_1{}^T(t_0)\ldots\mathbf{H}_{N-1}^T(t_0)\,\mathbf{H}_N{}^T(t)$ and the objectivity principle should be considered for $\mathbf{Q}(t) = \mathbf{R}^T(\mathbf{X}_0, t)$. The equations for the production rates of the inelastic deformations are here convenient to use in a form equivalent to (2.3)

$$\mathbf{\Phi}_\beta'\left[\frac{\partial}{\partial t}(\mathbf{P}_N\mathbf{P}_{N-1}\ldots\mathbf{P}_\beta), \mathbf{F}, \mathbf{P}_\alpha, \theta, \gamma\right] = 0$$

$\alpha, \beta = 1, 2, \ldots, N$.

The sufficiency of (2.14) and (2.15) for the satisfaction of the objectivity principle is evident. To prove the sufficiently of (2.14) and (2.15) for the satisfaction of (2.8), the fact that (2.14) is a narrowing of systems analogous to (2.12) and (2.13), whose sufficiency is clearly seen, should be used.

We will consider constraints on the governing relationships of viscoelastic bodies of relaxation type that are imposed by the second law of thermodynamics

$$-\rho A^{\cdot} - \rho\eta\theta^{\cdot} + \mathrm{tr}\,(\mathbf{T}\mathbf{F}^{-1T}\mathbf{F}^{\cdot}) + \theta^{-1}\mathbf{q}\nabla\theta \geqslant 0 \tag{2.17}$$

Assuming that the functions $A^+, \mathbf{T}^+, \eta^+, \mathbf{q}^+$ and $\mathbf{\Psi}_\alpha$ are defined and continuously differentiable in an open simply-connected domain of the variables $\{\mathbf{F}, \mathbf{V}_\alpha, \theta, \gamma\}$ it can be shown that

$$A = A^+(\mathbf{U}, \mathbf{V}_\alpha, \theta), \quad \eta = -\frac{\partial A^+}{\partial\theta} \tag{2.18}$$

$$\mathbf{T} = \rho\frac{\partial A^+}{\partial\mathbf{F}}\mathbf{F}^T = \rho\mathbf{R}\frac{\partial A^+}{\partial\mathbf{U}}\mathbf{R}^T$$

$$\delta = -\sum_{\alpha=1}^{N}\mathrm{tr}\left(\frac{\partial A^+}{\partial\mathbf{V}_\alpha}\mathbf{\Psi}_\alpha\right) + \frac{1}{\rho\theta}\mathbf{q}\nabla\theta \geqslant 0$$

It is seen from (2.18) that the inequality (2.17) for the materials being studied will result in partial splitting of the temperature gradient and strain-stress effects, the entropy and the free energy are independent of $\nabla\theta$.

For the case of an ideal viscoelastic body, the relationships (2.18) allow of greater simplification. To execute them we note that the non-degenerate tensor $\mathbf{B}$ is non-symmetric in the general case. Consequently, $\mathbf{B} = \mathbf{QS}$, where $\mathbf{Q}$ is an orthogonal, and S a symmetric positive-definite tensor. For them the relation

$$\mathbf{Q} = \mathbf{R}^T\mathbf{R}_e\mathbf{H}, \quad \mathbf{S} = \mathbf{H}^T\mathbf{U}_e\mathbf{H}, \quad \mathbf{H} = \mathbf{H}_N\mathbf{H}_{N-1}\ldots\mathbf{H}_1$$

follow from (1.4) and (1.5) and the uniqueness of the polar expansion.

The free energy density A^+ as a function of S, is independent of $\mathbf{Q}$.

To show this we consider the particle deformation process $\mathbf{X} = \mathbf{X}_0$ for $t \geqslant t_0$ for which all the arguments of the state are unchanged, except $\mathbf{Q}$. Then

$$A^{\cdot} = \frac{\partial A^+}{\partial B_{ab}}\frac{\partial B_{ab}}{\partial Q_{ij}}Q_{ij}^{\cdot} = \frac{\partial A^+}{\partial B_{ab}}\delta_{ai}Q_{ij}^{\cdot}Q_{jp}^{\ T}B_{pb}$$

Since

$$\mathbf{T} = \rho\mathbf{R}\frac{\partial A^+}{\partial\mathbf{B}}\mathbf{B}^T\mathbf{R}^T$$

then by virtue of the symmetry of the Cauchy stress tensor

$$\rho A^{\cdot} = \mathrm{tr}\,[(\mathbf{R}^T\mathbf{T}\mathbf{R})\,\mathbf{\Omega}] \equiv 0, \quad \mathbf{\Omega} = \mathbf{Q}^{\cdot}\mathbf{Q}^T$$

Here and henceforth the subscript notation is used in Cartesian orthogonal coordinates in those relationships where tensors of the third and higher ranks appear.

If isotropy of the function A^+ is used, then the dependence on S can be represented in the form

$$A = A_1(\mathbf{e}, \Delta_\alpha, \theta), \quad \mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{F}^{-1T}\mathbf{V}\mathbf{V}^T\mathbf{F}^{-1}) \tag{2.19}$$

$$\mathbf{V} = \mathbf{V}_1\mathbf{V}_2\ldots\mathbf{V}_N, \quad \mathbf{V} \neq \mathbf{V}^T$$

from which it follows that

$$\mathbf{T} = \rho\,(\mathbf{I} - 2\mathbf{e})\,\partial A_1/\partial\mathbf{e} \tag{2.20}$$

3. **Example.** We will consider the important practical case of an ideal viscoelastic medium with small temperature gradients and small elastic deformations that build up in a background of large irreversible deformations and large rotations. For simplicity, we will limit ourselves to the case when $N = 1$.

If the symmetric part $\mathbf{U}_e$ of the gradient $\mathbf{E}$ tends to one, then $\mathbf{F} = \mathbf{R}_e\mathbf{U}_e\mathbf{H}\mathbf{W} \to \mathbf{R}_e\mathbf{H}\mathbf{W}$, where $\mathbf{W} \equiv \mathbf{W}_1$, $\mathbf{H} \equiv \mathbf{H}_1$ and, therefore, $\mathbf{U} \to \mathbf{W}, \mathbf{R} \to \mathbf{R}_e\mathbf{H}$. Introducing the tensor $\boldsymbol{\varepsilon}$ of small elastic deformations such that

$$\mathbf{U} = \mathbf{W} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T, \quad \|\boldsymbol{\varepsilon}\| \ll 1 \tag{3.1}$$

we find that the following tensors are also small

$$\mathbf{B}-\mathbf{I}=\boldsymbol{\varepsilon}\mathbf{W}^{-1},\quad \mathbf{B}^{-1}-\mathbf{I}=\mathbf{W}^{-1}\boldsymbol{\varepsilon}$$
$$\mathbf{e}={}^1/_2\mathbf{R}(\boldsymbol{\varepsilon}\mathbf{W}^{-1}+\mathbf{W}^{-1}\boldsymbol{\varepsilon})\mathbf{R}^T \tag{3.2}$$

The expansion of the free energy density (2.19) in a Taylor series in the neighbourhood of $\mathbf{e}=0$ to second-order term accuracy, taking the isotropy A_1 and the relation (2.20) into account can be written in the form

$$\rho_* A_1=\rho_* A_0(\theta,\Delta)-p_*(\theta,\Delta)I_1+{}^1/_2\lambda(\theta,\Delta)I_1^2+\mu(\theta,\Delta)I_2 \tag{3.3}$$
$$\Delta\equiv\Delta_1,\ I_1=\operatorname{tr}\mathbf{e},\ I_2=\operatorname{tr}\mathbf{e}^2$$

where $\rho_*=\rho_0/\Delta$ is the density of the material in the IC $\varkappa_1$, $\mathbf{T}_*=-p_*\mathbf{I}$ is the Cauchy stress tensor in this configuration, λ,μ are scalars possibly dependent on θ and Δ. It follows from (3.3) and (2.20) that

$$\mathbf{T}=-(1-I_1)p_*\mathbf{I}+\lambda I_1\mathbf{I}+2(\mu+p_*)\mathbf{e}$$
$$\eta=-\frac{\partial A_0}{\partial\theta}+\frac{1}{\rho_*}\frac{\partial p_*}{\partial\theta}I_1 \tag{3.4}$$

We consider the equation for the rate of production of the inelastic deformations

$$\mathbf{W}^{\cdot}\mathbf{W}^{-1}=\boldsymbol{\Psi}(\mathbf{B},\Delta,\theta,\mathbf{B}^{-1T}\nabla\theta)$$

For small temperature gradients it follows from the symmetry of the isotropic function

$$\varphi(\mathbf{B},\mathbf{W},\theta,\nabla\theta)\equiv\boldsymbol{\Psi}(\mathbf{B},\Delta,\theta,\mathbf{B}^{-1T}\nabla\theta)\mathbf{W}$$

that $\partial\varphi/\partial(\mathbf{B}^{-1T}\nabla\theta)\,|_{\nabla\theta=0}\equiv 0$.

This means that $\boldsymbol{\Psi}=\boldsymbol{\Psi}(\mathbf{B},\Delta,\theta)$ in a linear approximation in $\nabla\theta$.

By virtue of the smallness of the elastic deformations defined by (3.1) and (3.2), we have the following functions continuously differentiable with respect to $\mathbf{B}$

$$\Psi_{ij}(\mathbf{B},\Delta,\theta)=\Psi_{ij}|_{e=0}+\left.\frac{\partial\Psi_{ij}}{\partial B_{ab}}\right|_{e=0}(B_{ab}-\delta_{ab})+O(\|e\|^2)$$

where it follows from the isotropy of second- and fourth-rank tensors

$$\Psi_{ij}|_{e=0}=\psi_0(\theta,\Delta)\delta_{ij}$$
$$\left.\frac{\partial\Psi_{ij}}{\partial B_{ab}}\right|_{e=0}=\frac{\delta_{ij}\delta_{ab}}{3\tau_\lambda(\theta,\Delta)}+\frac{\delta_{ia}\delta_{jb}}{\tau_\mu(\theta,\Delta)}+\frac{\delta_{ib}\delta_{ja}}{\tau_\nu(\theta,\Delta)}$$

Taking into account that $\operatorname{tr}(\mathbf{B}-\mathbf{I})=\operatorname{tr}(\boldsymbol{\varepsilon}\mathbf{W}^{-1})=\operatorname{tr}\mathbf{e}=I_1$, while $1/\tau_\nu\equiv 0$ follows from the symmetry of $\mathbf{W}^{\cdot}$ we find the final form of the equation for the rate of production of the inelastic deformations of an ideal viscoelastic material with small elastic deformations and small temperature gradients

$$\mathbf{W}^{\cdot}=\left\{\psi_0(\theta,\Delta)+\frac{I_1}{3\tau_\lambda(\theta,\Delta)}\right\}\mathbf{W}+\frac{1}{\tau_\mu(\theta,\Delta)}(\mathbf{U}-\mathbf{W}) \tag{3.5}$$

We analogously obtain the Fourier law $\mathbf{q}=k_0(\theta,\Delta)\nabla\theta$ with a scalar thermal conductivity possibly dependent on the temperature and the volume inelastic deformation for the heat flux in the approximation under consideration.

As an illustration, we calculate the velocity of propagation of a weak discontinuity wave in the material under consideration. In addition, we assume that the process is isothermal and proceeds at the temperature θ_0 of the RC, and the IC $\varkappa_1$ is an unloaded configuration with zero stresses, $\Delta\equiv 1$, i.e., there are no bulk inelastic deformations. Then $\psi_0=0$, $\tau_\lambda^{-1}+\tau_\mu^{-1}=0$, $\lambda,\mu,\tau_\mu=\text{const}$, and (3.4) and (3.5) reduce to the form

$$\mathbf{T}=\lambda I_1\mathbf{I}+2\mu\mathbf{e},\quad \mathbf{W}^{\cdot}=\frac{1}{\tau}\left\{\mathbf{U}-\left(1+\frac{1}{3}I_1\right)\mathbf{W}\right\},\quad \tau\equiv\tau_\mu \tag{3.6}$$

From system (3.6) we obtain the following differential equation ($\mathbf{T}'=\mathbf{T}-{}^1/_3(\operatorname{tr}\mathbf{T})\mathbf{I}$ is the Cauchy stress tensor deviator):

$$\mathbf{T}^{\cdot}+\tau^{-1}\mathbf{T}'=\lambda\operatorname{tr}(\nabla\mathbf{v}+\mathbf{e}\nabla\mathbf{v})\mathbf{I}+\mu(\nabla\mathbf{v}+\nabla\mathbf{v}^T+\mathbf{e}\nabla\mathbf{v}+\nabla\mathbf{v}^T\mathbf{e}) \tag{3.7}$$

It is hence seen that when the deformation process halts at the time $t=t_0$, i.e., for $\nabla\mathbf{v}=0$, the global part of the stress tensor will remain invariant while the deviator relaxes according to the law

$$\mathbf{T}'(t)=\mathbf{T}'(t_0)\exp(-(t-t_0)/\tau)$$

Appending the equation of motion of (3.7)

$$\rho\mathbf{v}^{\cdot}-\operatorname{div}\mathbf{T}=\rho\mathbf{b} \tag{3.8}$$

and taking into account that $\mathbf{e}$ is expressed in a linear manner in terms of $\mathbf{T}$ from the first relationship in (3.6), we obtain a closed system of equations in the variables $\mathbf{v},\mathbf{T}$.

Let $\varphi(x, t) = 0$ be the equation of the weak discontinuity surface $D = -|\nabla\varphi|^{-1}\partial\varphi/\partial t$, $\mathbf{n} = |\nabla\varphi|^{-1}\nabla\varphi$ the propagation velocity and unit normal to the surface. Let $c = D - \mathbf{v}\cdot\mathbf{n}$ be the propagation velocity of the surface relative to the particles of the medium, and $\mathbf{V} = [\partial\mathbf{v}/\partial n]_-^+$, $\mathbf{S} = [\partial\mathbf{T}/\partial n]_-^+$, $\mathbf{E} = [\partial\mathbf{e}/\partial n]_-^+$ are jumps in the normal derivatives of the velocity vector, the Cauchy stress tensor, and the elastic deformations tensor. For $c \neq 0$ it follows from (3.7) and (3.8) that

$$a\mathbf{V} - b(\mathbf{V} + \delta M)\mathbf{n} - \mu\mathbf{e}\mathbf{V} = 0 \tag{3.9}$$

$$a = \rho c^2 - \mu,\quad b = \lambda + \mu,\quad \delta = \|\mathbf{e}\| = (\operatorname{tr}\mathbf{e}^2)^{1/2}$$
$$\mathbf{m} = \mathbf{en}/\delta,\quad N = \mathbf{V}\cdot\mathbf{n},\quad M = \mathbf{V}\cdot\mathbf{m}$$

In the zeroth approximation corresponding to $\mathbf{e} = 0$, Eq.(3.9) takes the form $a_0\mathbf{V} + bN\mathbf{n} = 0$, from which $a_0 = b$ and $\mathbf{V} = N\mathbf{n}$ or $N = 0$, $a_0 = 0$ follow.

The first case describes a longitudinal wave with amplitude vector of the weak discontinuity of the velocity collinear with $\mathbf{n}$ so that

$$\rho c_0^2 = \lambda + 2\mu,\quad \mathbf{V} = N\mathbf{n}$$

The second case corresponds to a transverse wave on which

$$\rho c_0^2 = \mu,\quad \mathbf{V}\cdot\mathbf{n} = 0$$

Let $\mathbf{l} = \mathbf{een}/\delta^2$, $|\mathbf{l}| = O(1)$, $L = \mathbf{l}\cdot\mathbf{n}$. If the vectors $\mathbf{n}, \mathbf{m}$ and $\mathbf{l}$ are linearly independent, then by convoluting (3.9) with $\mathbf{n}, \mathbf{m}$ and $\mathbf{l}$ and using the Hamilton-Cayley theorem, we arrive at a system of three linear homogeneous equations

$$\begin{aligned}
&(b - a)N + \delta(b + \mu)M = 0\\
&b(\mathbf{m}\cdot\mathbf{n})N + [\delta b(\mathbf{m}\cdot\mathbf{n}) - a]M + \delta\mu L = 0\\
&[b(\mathbf{l}\cdot\mathbf{n}) + \delta\mu J_3]N + \delta[b(\mathbf{l}\cdot\mathbf{n}) - \mu J_2]M - [\delta\mu J_1 - a]L = 0\\
&J_1 = \delta^{-1}\operatorname{tr}\mathbf{e},\quad J_2 = {}^1/_2[J_1^2 - \delta^{-2}\operatorname{tr}\mathbf{e}^2],\quad J_3 = \delta^{-3}\det\mathbf{e}
\end{aligned} \tag{3.10}$$

From the fact that the determinant of the coefficient matrix of system (3.10) vanishes when $a_0 \neq 0$, it follows in a linear approximation in δ that $a = a_0 + \delta a_1 = b + \delta(\lambda + \mu)(\mathbf{m}\cdot\mathbf{n})$ or

$$= (\lambda + 2\mu)(1 + \mathbf{n}\cdot\mathbf{en}) \tag{3.11}$$

The weak discontinuity waves propagating at the velocity (3.11) can be called quasi-longitudinal waves since the polarization of such waves is determined by the formula

$$\mathbf{V} = V_0(\mathbf{n} + \delta\mu b^{-1}\mathbf{m})$$

When $a_0 = 0$ we have $a = \delta a_1 + O(\delta^2)$. The equation

$$a_1^2 - \mu(J_1 - \mathbf{m}\cdot\mathbf{n})a_1 + \mu^2(J_2 - J_1\mathbf{m}\cdot\mathbf{n} + \mathbf{l}\cdot\mathbf{n}) = 0$$

is obtained for a_1, whose solution is

$$a_1 = \frac{\mu}{2}\left\{J_1 - \mathbf{m}\cdot\mathbf{n} \pm (J_1 + \mathbf{m}\cdot\mathbf{n})\left[1 - \frac{4(J_2 + \mathbf{l}\cdot\mathbf{n})}{(J_1 + \mathbf{m}\cdot\mathbf{n})^2}\right]^{1/2}\right\} \tag{3.12}$$

In order to show the non-negativity of the radicand in the solution (3.12), we use an orthonormalized basis that agrees with the triplet of eigenvectors of the tensor $\mathbf{e}$. Then the non-negativity condition reduces to the inequality

$$\begin{aligned}
&\Big\{\sum_{i=1}^3(1 + n_i^2)e_i\Big\}^2 + 2\sum_{i=1}^3(1 - 2n_i^2)e_i^2 - 2\Big(\sum_{i=1}^3 e_i\Big)^2 =\\
&(1 - n_1^2)^2e_1^2 + (1 - n_2^2)^2e_2^2 + (1 - n_3^2)^2e_3^2 + 2(n_1^2n_2^2 - n_3^2)e_1e_2 +\\
&2(n_1^2n_3^2 - n_2^2)e_1e_3 + 2(n_2^2n_3^2 - n_1^2)e_2e_3 \geqslant 0
\end{aligned}$$

The symmetric quadratic form under consideration will be non-negative if and only if all the minors of the determinants symmetric with respect to the main diagonal of the coefficient matrix are non-negative. We find by a direct calculation that all the second-order minors equal $4n_1^2n_2^2n_3^2 \geqslant 0$, and the third order minor equals zero. Therefore, the quadratic form is non-negative and the propagation velocities under consideration

$$\rho c^2 = \mu + \frac{\mu}{2}\left\{\operatorname{tr}\mathbf{e} - \mathbf{n}\cdot\mathbf{en} \pm (\operatorname{tr}\mathbf{e} + \mathbf{n}\cdot\mathbf{en})\left[1 - \frac{4(J_2\|\mathbf{e}\|^2 + \mathbf{en}\cdot\mathbf{en})}{(J_1\|\mathbf{e}\| + \mathbf{n}\cdot\mathbf{en})^2}\right]^{1/2}\right\}$$

are real quantities. In contrast to the zeroth approximation, these velocities are not multiple for $\|\mathbf{e}\| \neq 0$. The polarization of such waves is determined by (3.10) in a unique manner, and in particular, is characterized by the fact that the vector of the weak discontinuity of the velocity has a normal component proprotional to $\delta = \|\mathbf{e}\|$. This property enables us to call the waves quasitransverse.

REFERENCES

1. TRUESDELL C. and NOLL W., Handbuch der Physik, Band /111/3. The non-linear field theories of mechanics. Springer, Berling, 1965.
2. IL'YUSHIN A.A. and POBEDRYA B.E., Principles of the Mathematical Theory of Thermovisco-elasticity. Nauka, Moscow, 1970.
3. LODGE A., Elastic Fluids /Russian translation/, Nauka, Moscow, 1969.

4. ASTARINA G. and MARRUCCI J., Principles of the Hydromechanics of Non-Newtonian Fluids /Russian translation/, Mir, Moscow, 1978.
5. GERMAIN P., Course in Mechanics of Continuous Media. General Theory /Russian translation/, Vysshaya Shkola, Moscow, 1983.
6. LEE E.H., Elastic-plastic deformation at finite strains. Trans. ASME, Ser. E., J. Appl. Mech., 36, 1, 1969.
7. FREUND L.B., Constitutive equations for elastic-plastic materials at finite strain. Intern. J. Solids and Struct., 6, 1970.
8. KONDAUROV V.I., On equations of elastic viscoplastic media with finite strains. Priklad. Mekhan. Tekh. Fiz., 4, 1982.
9. LUR'E A.I., Non-linear Theory of Elasticity. Nauka, Moscow, 1980.
10. NOLL W., Proof of maximality of the orthogonal group in the unimodular group. Arch. Rat. Mech. and Analysis, 18, 1, 1965.

Translated by M.D.F.

PMM U.S.S.R.,Vol.49,No.5,pp.615-619,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

DYNAMIC PROBLEMS FOR A PLANE AND CYLINDRICAL VISCOELASTIC LAYER PARTIALLY ADHERENT TO A STIFF RING*

S.I. GRITSENKO

The plane problem is examined of the shear-vibration of an infinite stiff viscoelastic layer covering that adheres partially to an undeformable cover-foundation: rigidly along a strip of width $2a$, and in frictionless contact outside this strip. In addition, an analogous axisymmetric problem is considered for a cylindrical viscoelastic layer. The layer is partially adherent to a ring along one surface: rigidly along a band of width $2a$ and without friction outside this band, and it is rigidly adherent to a ring vibrating in the axial direction along the other surface.

Mixed boundary value problems reduce to the solution of an integral equation of the first kind which reduces, in turn, to an infinite system of linear algebraic equations. Certain results are presented of a numerical solution of the problems posed. Solutions are compared for the visco-elastic and corresponding elastic problems. The efficiency of two methods of solving the integral equation, reduction to an infinite system and approximation of its kernel, is compared for the latter problem.

1. We examine the plane problem of steady vibrations of a viscoelastic layer $0 \leqslant z \leqslant h$, $|x| < \infty$ lying on an undeformable foundation $z = 0$. The layer is rigidly aherent to the foundation along the strip $|x| \leqslant a$ of width $2a$ and makes friction-free contact outside this strip. Along the whole upper boundary $z = h$ the layer is rigidly aherent to an undeformable covering vibrating in a tangential direction (problem A). The boundary conditions of problem A have the form

$$\begin{aligned}
&u_x(x, h, t) = U_0 e^{-i\omega t}, \quad u_z(x, h, t) = 0\\
&u_z(x, 0, t) = 0, \quad |x| < \infty\\
&u_x(x, 0, t) = 0, \quad |x| \leqslant a; \ \tau_{xz}(x, 0, t) = 0, \quad |x| > a
\end{aligned}$$

In addition to problem A, we consider an analogous axisymmetric problem for a viscoelastic cylindrical layer $R_0 \leqslant r \leqslant R_h$, $|x| < \infty$ (the third cylindrical coordinate z is replaced here by x for uniformity in the subsequent calculations). The cylindrical layer is rigidly adherent to a fixed undeformable ring along a strip $|x| \leqslant a$ of width $2a$ at its inner surface $r = R_0$ and abuts it without friction outside the strip. Along the whole external surface $r = R_h$ the cylindrical layer is rigidly adherent to an undeformable ring vibrating in the axial direction (problem B1). The boundary conditions of problem B1 have the form

$$\begin{aligned}
&u_x(R_h, x, t) = U_0 e^{-i\omega t}, \quad u_r(R_h, x, t) = 0\\
&u_r(R_0, x, t) = 0, \quad |x| < \infty\\
&u_x(R_0, x, t) = 0, \quad |x| \leqslant a; \ \tau_{rx}(R_0, x, t) = 0, \quad |x| > a
\end{aligned}$$

*Prikl.Matem.Mekhan.,49,5,801-807,1985